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N-Tuples in the Class $\mathbb{A}^{(N)}$ and Factorization Properties

Frédéric Jaëck

U.F.R. Mathématiques et Informatique, Université Bordeaux I, 351, cours de la Libération, 33405 Talence Cedex, France
E-mail: jaeck@math.u-bordeaux.fr

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The main purpose of this paper is to exhibit N-tuples of commuting operators ($N \geqslant 2$) in the class $\mathbb{A}^{(N)}$ (i.e., with an isometric w$*$-continuous $H^\infty(\mathbb{D}^N)$ calculus) without the approximate factorization property. © 2002 Elsevier Science (USA)

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Let $\mathbb{T}$ denote the unit circle in the complex plane, $\mathscr{B}$ the Borel σ-algebra on $\mathbb{T}^N$, and m_N the normalized Lebesgue measure.

For a given normalized Borel measure μ on $\mathbb{T}^N$, we denote by $P^\infty(\mu)$ the w$*$-closed sub-algebra of $L^\infty(\mathbb{T}^N, \mu)$ generated by the polynomials, equipped with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{z \in \mathbb{T}^N} |f(z)|$.

As usual $H^\infty(\mathbb{D}^N)$ will denote the Banach algebra of all bounded analytic functions on $\mathbb{D}^N$ and will be identified in the usual way with $H^\infty(\mathbb{T}^N) = P^\infty(m_N)$.

The space $P^\infty(\mu)$ is isometrically isomorphic to the norm dual of the predual Banach space $\mathscr{Q}_\mu = L^1(\mathbb{T}^N, \mu)/{}^\perp P^\infty(\mu)$, where

$$^\perp P^\infty(\mu) = \left\{ \ell \in L^1(\mathbb{T}^N, \mu);\ \int_{\mathbb{T}^N} \ell u \, d\mu = 0\ \forall u \in P^\infty(\mu) \right\}.$$

In general, for any $\ell \in L^1(\mathbb{T}^N, \mu)$ we will denote by $[\ell]_\mu$ the image of ℓ in the quotient space $\mathscr{Q}_\mu$.

For a given Hilbert space $\mathscr{D}$ and a measure μ on $\mathbb{T}^N$, we denote by $L^2(\mathbb{T}^N, \mu, \mathscr{D})$ the Hilbert space of all measurable square integrable functions $f\colon \mathbb{T}^N \to \mathscr{D}$ with the scalar product

$$\langle f \mid g \rangle = \int_{\mathbb{T}^N} \langle f(\zeta) \mid g(\zeta) \rangle_{\mathscr{D}} \, dm_N(\zeta).$$

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For $f, g \in L^2(\mathbb{T}^N, \mu, \mathcal{D})$, the function $f \cdot g$ is defined by setting $f \cdot g(\zeta) = \langle f(\zeta) \mid g(\zeta) \rangle_{\mathcal{D}}$ for almost every $\zeta \in \mathbb{T}^N$.

We denote by $U = (U_1, \ldots, U_N)$ the N-tuple of multiplications by the independent variables on $L^2(\mathbb{T}^N, \mu, \mathcal{D})$: $U_i f(z_1, \ldots, z_N) = z_i f(z_1, \ldots, z_N)$.

For any multi-index $k = (k_1, \ldots, k_N) \in \mathbb{N}^N$ we will write $U^k = U_1^{k_1} \cdots U_N^{k_N}$.

From now on $\mathcal{H}$ will be a semi-invariant space for U. We denote by $T = (T_1, \ldots, T_N)$ the compression of U to $\mathcal{H}$, which means, by definition: $T^k f = \mathrm{P}_{\mathcal{H}} U^k f$, for $k \in \mathbb{N}^N$ and $f \in L^2(\mathbb{T}^N, \mu, \mathcal{D})$ (such a T is a model for N-tuples possessing a unitary dilation).

We recall that the algebra of all bounded operators $\mathcal{L}(\mathcal{H})$ can be realized as the norm-dual of the Banach space of all trace-class operators $\mathcal{C}^1(\mathcal{H})$. The dual algebra generated by T (i.e., the smallest w$*$-closed algebra containing all the T_is) will be denoted by $\mathcal{A}_T$ and can be identified with the dual of $\mathcal{Q}_T = \mathcal{C}^1(\mathcal{H}) / {}^{\perp}\mathcal{A}_T$, where ${}^{\perp}\mathcal{A}_T = \{C \in \mathcal{C}^1(\mathcal{H});\ tr(AC) = 0\ \forall A \in \mathcal{A}_T\}$.

We recall some properties for the contraction T. For any given $x, y \in \mathcal{H}$, we denote by $x \square y$ the element of $\mathcal{Q}_T$ defined by $\langle A \mid x \square y \rangle = (Ax \mid y)$ $\forall A \in \mathcal{A}_T$. Given two cardinal numbers $1 \leqslant m, n \leqslant \aleph_0$, we say that T has property $(\mathbb{A}_{m,n})$ (or $(\mathbb{A}_n)$ for short if $m = n$) if for any collection $\{L_{p,q};\ 1 \leqslant p < m, 1 \leqslant q < n\}$ of elements in $\mathcal{Q}_T$ there exist sequences $(x_k)_k$ and $(y_k)_k$ in $\mathcal{H}$ such that

$$L_{p,q} = x_p \square y_q \quad 1 \leqslant p < m \qquad \text{and} \qquad 1 \leqslant q < n.$$

Finally, we say that T is in the class $\mathbb{A}^{(N)}$ (or equivalently that the algebra $\mathcal{A}_T$ is of type $\mathbb{A}$) if there exists an isometric w$*$-continuous morphism Φ: $H^\infty(\mathbb{D}^N) \longrightarrow \mathcal{A}_T$, which extends the polynomial calculus.

Now returning to functional properties, we say that a subspace $\mathcal{H} \subset L^2(\mathbb{T}^N, \mu, \mathcal{D})$ has the approximate factorization property if for every function $\ell \in L^1(\mathbb{T}^N, \mu)$ and every $\epsilon > 0$, there exist vectors x, y in $\mathcal{H}$ such that

$$\|\ell - x \cdot y\|_1 < \epsilon.$$

In the case when $N = 1$, Bercovici proved the following result:

THEOREM 1.1 ([1] Theorem 10). *Assume that T is in the class $\mathbb{A} (= \mathbb{A}^{(1)})$, then for every given $f \in L^1(\mathbb{T}, m_1)$, every $\epsilon > 0$, and every collection $\zeta_1, \ldots, \zeta_p \in L^2(\mathbb{T}, m_1, \mathcal{D})$, there exist x and y in $\mathcal{H}$ such that*

(1) $\|x\|, \|y\| \leqslant \|f\|^{1/2}$

(2) $\langle x \mid \zeta_j \rangle = \langle y \mid \zeta_j \rangle = 0$ $j = 1, \ldots, p$

(3) $\|f - x \cdot y\|_1 < \epsilon$.

In particular, the space $\mathscr{H}$ has the approximate factorization property.

Property $(\mathbb{A}_1)$, which is the weakest of the above considered properties, is known to have relations to invariant subspaces and dilation theory.

Again in the case when $N=1$, Bercovici and Chevreau independently proved the following stimulating result:

THEOREM 1.2 [1, 3]. *Let us assume that $T \in \mathscr{L}(\mathscr{H})$ is in the class $\mathbb{A}(=\mathbb{A}^{(1)})$. Then the contraction T has property $(\mathbb{A}_1)$.*

In the proof of this result, Theorem 1.1 appeared to be a crucial tool.

In this paper we construct a subspace $\mathscr{H}$ that is reducing for the N-tuple of unitary operators U, such that $U_{|H}$ is in the class $\mathbb{A}^{(N)}$, $N \geqslant 2$, and such that $\mathscr{H}$ does not have the approximate factorization property.

This is based on the study of the possible values of the moduli of the elements in $P^\infty(\mu)$ and on the existence of related Borel sets in $\mathbb{T}^N$.

2. ALGEBRAS OF TYPE $\mathbb{A}$ THAT DO NOT HAVE THE APPROXIMATE FACTORIZATION PROPERTY

We first recall a few definitions relative to subsets of $\mathbb{T}^N$ and we give a simple construction of some sets that will be useful later on.

We recall that a set $C \subset \mathbb{T}^N$ is said to be circular if for any $\zeta \in \mathbb{T}$ and any $c \in C$ we have $\zeta c \in C$. The existence of circular subsets of $\mathbb{T}^N$ with empty interior and positive measure is well known, but for the sake of completeness, we give an easy construction of such a set. We start with $N=2$ but it immediately extends to any larger $N \geqslant 2$.

We first consider a Cantor-type set $\sigma \subset \mathbb{T}$, such that σ is a compact set of empty interior with $m_1(\sigma) > 0$. Then we consider the set $\Sigma = \{(z\zeta, \zeta) / z \in \sigma,\ \zeta \in \mathbb{T}\}$. To show that Σ possesses the desired properties, we consider the application θ defined on $\mathbb{T}^2$ by $\theta(w_1, w_2) = (w_1 w_2, w_2)$. Since θ is continuous, the set $\Sigma = \theta(\sigma \times \mathbb{T})$ is compact of empty interior. This set is circular by construction and we can compute

$$m_2(\Sigma) = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \chi_\Sigma(w_1, w_2)\, dm_1(w_1) \right) dm_1(w_2) = m_1(\sigma) > 0.$$

We now turn to the study of properties shared by $H^\infty(\mathbb{T}^N)$ functions.

For $w \in \mathbb{T}^N$ and $u \in H^\infty(\mathbb{D}^N)$, we denote by u_w the element of $H^\infty(\mathbb{D})$ defined by $u_w(\lambda) = u(\lambda w)$, for $\lambda \in \mathbb{D}$.

The subset of $\mathbb{T}^N$ where the radial limits of u exist will be denoted by $\mathscr{RL}(u)$ and we have $m_N(\mathscr{RL}(u)) = 1$. Now for a fixed point w in $\mathbb{T}^N$, and

$z \in \mathcal{RL}(u_w) \subseteq \mathbb{T}$, $\lim_{r\to 1} u_w(rz)$ exists and then at any point $wz \in \mathbb{T}^N$, $z \in \mathcal{RL}(u_w)$, the radial limit for u exists.

For $w \in \mathbb{T}^N$, $z \in \mathcal{RL}(u_w)$ we have $(u_w)^*(z) = u^*(zw)$.

We now recall a well-known property of the moduli of the elements in $H^\infty(\mathbb{T}^N)$; we give the proof for the sake of completeness (see [5]):

LEMMA 2.1. *Given* $u \in H^\infty(\mathbb{D}^N)$ $w \in \mathbb{T}^N$, *we consider the following function defined everywhere on* $\mathbb{T}^N$*:*

$$G_u\colon \mathbb{T}^N \longrightarrow [0; +\infty[$$
$$w \longrightarrow \operatorname*{ess\,sup}_{z\in\mathbb{T}} |u^*(zw)|.$$

Then G_u *is lower semi-continuous on* $\mathbb{T}^N$.

Proof. For $w \in \mathbb{T}^N$, thanks to the maximum modulus theorem we have $G_u(w) = \sup_{0<r<1} \sup_{z\in\mathbb{T}} |u(rzw)|$. Now the function $w \to |u(rzw)|$ is continuous for all r and z, which leads to the conclusion. ∎

We now recall a lemma that is useful to work with the "slice functions" introduced earlier, and with circular sets (see [5], p. 45, L.3.3.2).

LEMMA 2.2. *If* $\psi \in L^1(\mathbb{T}^N)$ *then we have*

$$\int_{\mathbb{T}^N} dm_N(w) \int_{\mathbb{T}} \psi(\lambda w)\, dm_1(\lambda) = \int_{\mathbb{T}^N} \psi(w)\, dm_N(w).$$

Proof. This is a straightforward consequence of the translation-invariance of m_N together with Fubini's theorem: the left-hand-side integral does not depend on λ, and we integrate it over $\mathbb{T}$. ∎

For a given $w \in \mathbb{T}^N$ and a given subset $E \subset \mathbb{T}^N$, we write $E(w) := \{\lambda \in \mathbb{T} / \lambda w \in E\}$. We are now ready to explore a bit further the properties of the boundary values of functions in $H^\infty(\mathbb{T}^N)$. The following proposition will be an essential tool in the sequel. We begin with a usefull observation:

LEMMA 2.3. *Let F be a given subset in* $\mathbb{T}^N$*. Then we have*

$$m_N(F) = 0 \Leftrightarrow m_1(F(z)) = 0 \textit{ for almost every } z \textit{ in } \mathbb{T}^N.$$

Proof. We use Lemma 2.2 with $\psi = \chi_F$, which gives

$$m_N(F) = \int_{\mathbb{T}^N} dm_N(w) \int_{\mathbb{T}} \chi_F(\lambda w)\, dm_1(\lambda) = \int_{\mathbb{T}^N} m_1(F(w))\, dm_N(w).$$

The result follows imediately from this line. ∎

PROPOSITION 2.1. *Let $\mathcal{U} \subset \mathbb{T}^N$ $(N \geqslant 2)$ be a dense open circular subset of $\mathbb{T}^N$ with $m_N(\mathcal{U}) < 1$. We consider $u \in H^\infty(\mathbb{D}^N)$, with $\|u\|_\infty = 1$ and $\epsilon \in (0, 1)$. Then we have*

$$m_N(\{\zeta \in \mathcal{U} \,/\, |u^*(\zeta)| > 1-\epsilon\}) \neq 0,$$

where the intersection is to be understood up to a m_N-null set.

Proof. Let us note that $\mathcal{F} = \mathbb{T}^N \backslash \mathcal{U}$, which is a circular closed set of empty interior, such that $m_N(\mathcal{F}) \neq 0$.

We will also set $A_\epsilon = \{\zeta \in \mathbb{T}^N / \ |u^*(\zeta)| < 1-\epsilon\}$.

Let us suppose on the contrary that $m_N(A_\epsilon \cap \mathcal{U}) = 0$; then there exists a set $E \subset \mathcal{U}$ such that $m_N(E) = 0$ and $A_\epsilon \subset \mathcal{F} \cup E$.

Thanks to Lemma 2.1, we know that the set $\mathcal{O} = \{w \in \mathbb{T}^N / \text{ess}\sup_{z \in \mathbb{T}} |u^*(zw)| > 1-\epsilon\}$ is an open set, and $\mathcal{O}' = \mathcal{O} \cap \mathcal{U}$ is open too.

Now $\forall w \in \mathcal{O}'$ we have $m_1(E(w)) > 0$, which together with the fact that $m_N(E) = 0$ and Lemma 2.3 gives $\mathcal{O}' = \varnothing$. This in turn gives $\mathcal{O} \subset \mathcal{F}$, which is of empty interior, and then $\mathcal{O} = \varnothing$.

Now thanks to Lemma 2.3, we know that $m_N(A_\epsilon) > 0$ implies that $m_1(A_\epsilon(z)) > 0$ for many $z \in \mathbb{T}^N$, which is incompatible with $\mathcal{O} = \varnothing$. ▮

In order to emphasize the main difference that arises when going from $N = 1$ to the higher dimensional case, we recall a fundamental tool developped by Bercovici, which makes the junction between localization and factorization properties:

THEOREM 2.1 (see [2]). *Let $\mathcal{H} \subset L^2(\mathbb{T}^N, \mu, \mathcal{D})$ be a closed subset. Assume that for every set of positive measure $\sigma \subset \mathbb{T}^N$, every $\epsilon > 0$ and every finite set of vectors $\zeta_1, \ldots, \zeta_p \in L^2(\mathbb{T}^N, \mu, \mathcal{D})$, there exists a vector $z \in \mathcal{H}$, $z \neq 0$ such that:*

(1) *z is essentially bounded,*

(2) $\|\chi_{\mathbb{T}^N \backslash \sigma} z\| < \epsilon \, \|\chi_\sigma z\|$,

(3) $\langle z, \zeta_k \rangle = 0$, $k = 1, \ldots, p$.

Then $\mathcal{H}$ has the approximate factorization property. More precisely, given $f \in L^1(\mathbb{T}^N, \mu)$ and $\epsilon > 0$, there exist vectors $x, y \in \mathcal{H}$ such that:

(i) $\|f - x \cdot y\|_1 < \epsilon$

(ii) $\|x\| \cdot \|y\| < \|f\|_1$,

We are now ready for the construction of the example of an N-tuple.

Let $\mathcal{U} \subset \mathbb{T}^2$ be a dense open circular set of $\mathbb{T}^2$ with $m_2(\mathcal{U}) < 1$. We consider the action of the elements of $H^\infty(\mathbb{T}^2)$ over the space $\mathcal{H} = L^2(\mathcal{U}, m_2)$.

The following example is an illustration of the essential difference that appears in the study of algebras generated by several commuting contractions.

THEOREM 2.2. *Let $\mathcal{U} \subset \mathbb{T}^2$ be a dense open circular set of $\mathbb{T}^2$ with $m_2(\mathcal{U}) < 1$. Let $U = (U_1, U_2)$ be the pair of multiplications by the independent variables acting on $L^2(\mathbb{T}^2, m_2)$, and let $T = (T_1, T_2)$ be the compression of U to (the reducing space) $\mathcal{H} = L^2(\mathbb{T}^2, \chi_{\mathcal{U}} m_2)$.*

Then the pair $T = (T_1, T_2)$ is in the class $\mathbb{A}^{(2)}$ but the space $\mathcal{H}$ does not possess the approximate factorization property.

Proof. The calculus Φ: $H^\infty(\mathbb{D}^2) \to \mathcal{A}_U$ is defined by $\Phi(h) f = hf$ and is obviously a w$*$-continuous contractive morphism extending the polynomial calculus. We now prove that Φ is an isometry.

Let us consider $u \in H^\infty(\mathbb{D}^2)$, with $\|u\|_\infty = 1$ and $\epsilon \in (0, 1)$. We note $\mathcal{W} = \{\zeta \in \mathcal{U} \,/\, |u^*(\zeta)| > 1 - \epsilon\}$. Thanks to Proposition 2.1, we know that $m_2(\mathcal{W}) > 0$. So, we can find $f \in L^2(\mathbb{T}^2, \chi_{\mathcal{U}} m_2)$ such that $f \neq 0$ and $\|uf\|_2 > 1 - \varepsilon$. For example, take

$$f = \frac{\chi_{\mathcal{W}}}{m_2(\mathcal{W})^{1/2}}.$$

Now it is easy to see that the function 1 cannot be written $f \cdot g$ with $f, g \in L^2(\mathbb{T}^2, \chi_{\mathcal{U}} m_2)$, and $\|1 - f \cdot g\| \geqslant \|\chi_{\mathbb{T}^N \setminus \mathcal{U}}(1 - f \cdot g)\| = m_N(\mathbb{T}^N \setminus \mathcal{U}) > 0$. Hence the space $\mathcal{H}$ does not have the approximate factorization property. ∎

Remark. In fact, it is possible to identify the predual of such algebras, and to show that the pair T has property $(\mathbb{A}_{1, \aleph_0})$, although $\mathcal{H}$ does not have the approximate factorization property.

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